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Recently several definitions of weaker forms of continuity have appeared in the literature. In this paper, a study is made of some of these definitions with special attention being given to the definition of a somewhat continuous function.

WEAKER FORMS OF CONTINUITY

by

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INTRODUCTION

This paper is concerned with a study of various weaker forms of continuity. The definitions originated in [1], [2], and [4].

In Chapter I, the definitions of several types of continuity are given and examples are given to show that these are actually weaker than continuity.

In Chapter II, our main definition, somewhat continuous function, is given and some elementary properties of this type of function are discussed.

Chapter III is a discussion of weakly equivalent topologies and the relationships of this idea to our somewhat continuous functions.

In Chapter IV, the definition of a somewhat open function is presented and some elementary properties of this type of function are discussed.

In Chapter V, properties that carry over under our weaker forms of continuity are discussed. Also, somewhat homeomorphisms are defined and discussed here.

CHAPTER I

WEAKER FORMS OF CONTINUITY

Definition 1. Let f be a mapping of a topological space (X,S) into a topological space (Y,T). f is called an open function if $N \in S$ implies that $f(N) \in T$.

Definition 2. Let f be a mapping of a topological space (X,S) onto a topological space (Y,T). f is called a feebly open function if $N \subset X$ with interior $(N) \neq \emptyset$, implies interior $(f(N)) \neq \emptyset$.

Example 1. Let $X = \{a,b,c\}$. Let $S = \{X,\emptyset,\{a,b\}\}$ and $T = \{X,\emptyset,\{a\}\}$ be topologies for X. Let $f:(X,S) \to (X,T)$ be defined by $f(x) = x$. Then f is feebly open but f is not open.

Definition 3. Let f be a mapping from a topological space (X,S) into a topological space (Y,T). f is called a continuous function if $U \in T$ implies $f^{-1}(U) \in S$.

Definition 4. Let f be a mapping of topological space (X,S) into a topological space (Y,T). f is said to be weakly continuous if for each point $x \in X$ and each open set V of $f(x)$ there exists an open set U of x such that $f(U) \subset \overline{V}$.

Example 2. Let $X = \{a,b,c\}$. Let $S = \{X,\emptyset,\{a,b\}\}$ and $T = \{X,\emptyset,\{a\}\}$ be topologies for X. Let $f:(X,S) \to (X,T)$ be defined by $f(x) = x$. Then f is weakly continuous, but f is not continuous.

Remark 1. It is clear that a continuous function is weakly continuous.

Definition 5. Let f be a mapping of a topological space (X,S) onto a topological space (Y,T). f will be called feebly continuous if $M \subset Y$ with interior $(M) \neq \emptyset$ implies interior $(f^{-1}[M]) \neq \emptyset$.

Example 3. Let $X = \{a,b,c\}$. Let $S = \{X,\emptyset,\{a\}\}$ and $T = \{X,\emptyset,\{a,b\}\}$ be topologies for X. Let $f:(X,S) \to (X,T)$ be defined by $f(x) = x$. Then f is feebly continuous, but f is not continuous.

Definition 6. Let f be a mapping from a topological space (X,S) onto a topological space (Y,T). f is called quasi-continuous at a point a in its domain if and only if for each open set G containing $f(a)$ and each open set U containing a, there is a non-empty open set $V \subseteq U \cap f^{-1}[G]$.

Example 4. Let $X = \{a,b,c\}$. Let $S = \{X,\emptyset,\{a\}\}$ and $T = \{X,\emptyset,\{a,b\}\}$ be topologies for X. Let $f:(X,S) \to (X,T)$ be defined by $f(x) = x$. Then f is quasi-continuous at b but f is not continuous at b.

Definition 7. Let f be a mapping of a topological space (X,S) onto a topological space (Y,T). f will be called almost continuous A on X if for every element V of T, $\overline{\text{interior}\,(f^{-1}[V])} \supset f^{-1}[V]$.

Example 5. Let $X = \{a,b,c\}$. Let $S = \{X,\emptyset,\{a\}\}$ and $T = \{X,\emptyset,\{a,b\}\}$ be topologies for X. Let $f:(X,S) \to (X,T)$ be defined by $f(x) = x$. Then f is almost continuous A but f is not continuous.

Remark 2. It is clear that if a function is continuous and onto, then the function is almost continuous A.

Definition 8. Let f be a mapping of a topological space (X,S) into a topological space (Y,T). f is said to be almost continuous B at a point $x \in X$ if for every open set M of $f(x)$, there is an open set N of x such that $f(N) \subset$ interior $(\overline{M})$. f is said to be almost continuous B on X if f is also almost continuous B at each point of X.

Example 6. Let U be the usual topology for the reals R. Let $T = \{V \subset R : V$ contains a set of the form (a,∞) where $a \in R\} \cup \emptyset$. Then T is a topology for R and $f:(R,U) \to (R,T)$ defined by $f(x) = x$ is almost continuous B but not continuous.

Remark 3. It is clear that if a function is continuous, then the function is almost continuous B.

CHAPTER II

SOMEWHAT CONTINUOUS FUNCTIONS

Definition 9. Let f be a mapping of a topological space (X,S) into a topological space (Y,T). f is said to be somewhat continuous provided that if U is an element of T and $f^{-1}(U)$ is non-empty, then there is an element V in S such that V is non-empty and $V \subset f^{-1}(U)$.

Example 7. Let $X = \{a,b,c\}$. Let $S = \{X,\emptyset,\{a\}\}$ and $T = \{X,\emptyset,\{a,b\}\}$ be topologies for X. Let $f:(X,S) \to (Y,T)$ be defined by $f(x) = x$. Then f is somewhat continuous but f is not continuous.

Remark 4. It is clear that every continuous function is somewhat continuous.

Theorem 1. If $f:(X,S) \to (Y,T)$ and $g:(Y,T) \to (Z,U)$ are somewhat continuous functions and $f(X)$ is dense in Y, then $g \circ f:(X,S) \to (Z,U)$ is somewhat continuous.

Proof: Let $W \in U$ such that $(g \circ f)^{-1}(W) \neq \emptyset$. Since $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$, $g^{-1}(W) \neq \emptyset$. Then there is an open set $A \in T$ such that $A \neq \emptyset$ and $A \subset g^{-1}(W)$. Since $f(X)$ is dense in (Y,T), $f(X) \cap A \neq \emptyset$ and so $f^{-1}(A) \neq \emptyset$. Therefore there is an open set $B \in S$ such that $B \neq \emptyset$ and $B \subset f^{-1}(A)$. Now $B \subset f^{-1}(A) \subset f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ and hence $g \circ f$ is somewhat continuous.

Theorem 2. If $f:(X,S) \rightarrow (Y,T)$ is somewhat continuous and $g:(Y,T) \rightarrow (Z,U)$ is continuous, then $g\circ f:(X,S) \rightarrow (Z,U)$ is somewhat continuous.

Proof: Let W be an open set in (Z,U) such that $(g\circ f)^{-1}(W) \neq \emptyset$. Since g is continuous, $g^{-1}(W)$ is open in (Y,T). Since $f^{-1}(g^{-1}(W)) \neq \emptyset$ and since f is somewhat continuous, there is an open set $B \in S$ such that $B \neq \emptyset$ and $B \subset f^{-1}(g^{-1}(W)) = (g\circ f)^{-1}(W)$. Hence $g\circ f$ is somewhat continuous.

Theorem 3. If $f:(X,S) \rightarrow (Y,T)$ is a function, then the following three conditions are equivalent:

(1) f is somewhat continuous

(2) If C is a closed subset of Y such that $f^{-1}(C) \neq X$, than there is a proper closed subset D of X such that $D \supset f^{-1}(C)$, and

(3) If M is a dense subset of X, then $f(M)$ is a dense subset of $f(X)$.

Proof: $(1) \rightarrow (2)$ Suppose f is somewhat continuous. Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y - C$ is open and $f^{-1}(Y-C) \neq \emptyset$. Since f is somewhat continuous, there is a $V \in S$ such that $V \neq \emptyset$ and $V \subset f^{-1}(Y-C)$. Now $X - V$ is closed and since $V \neq \emptyset$, $X - V \neq X$. Let $x \in f^{-1}(C)$, then $f(x) \in C$, and thus $f(x) \notin Y - C$. Therefore $x \notin f^{-1}(Y-C)$ and thus $x \notin V$. Therefore $x \in X - V$. Hence $D = X - V \supset f^{-1}(C)$.

(2) $\Rightarrow$(3) Suppose f has property (2). Let M be a dense subset of X. Let U be a non-empty open subset of $f(X)$ under the relative topology. Then there is an element $V \in T$ such that $U = V \cap f(X)$. Therefore $Y - V$ is closed and since $f^{-1}(V) \neq \emptyset$, $f^{-1}(Y-V) \neq X$. Thus there is a proper closed subset D of X such that $D \supset f^{-1}(Y-V)$. Since $X - D$ is a non-empty open set and M is a dense subset of X, $X - D$ contains a point p of M. Since $p \notin D$, $p \notin f^{-1}(Y-V)$ and thus $f(p) \notin Y - V$ and $f(p) \in V$. Therefore $f(p) \in V \cap f(X) = U$. Hence U contains the point $f(p)$ of $f(M)$ and thus $f(M)$ is dense in $f(X)$.

(3) $\Rightarrow$(1) Suppose f has property (3). Let $U \in T$ such that $f^{-1}(U) \neq \emptyset$. Suppose $f^{-1}(U)$ does not contain a non-empty open subset of X. Then every non-empty open subset of X will intersect $X - f^{-1}(U)$ and thus $X - f^{-1}(U)$ is dense in X. Therefore $f[X - f^{-1}(U)]$ is dense in $f(X)$. Let $x \in f[X-f^{-1}(U)]$. Then there is a $y \in X - f^{-1}(U)$ such that $f(y) = x$. Thus $y \notin f^{-1}(U)$ and $f(y) \notin U$. Therefore $x = f(y) \in f(X) - U$. Hence $f[X - f^{-1}(U)] \subset f(X) - U$. Thus $U \cap f(X)$ is a non-empty open subset of $f(X)$ which contains no point of $f[X - f^{-1}(U)]$ which is impossible since $f[X - f^{-1}(U)]$ is dense in $f(X)$. Therefore $f^{-1}(U)$ contains a non-empty open subset of X and f is somewhat continuous.

Theorem 4. If $f:(X,S) \to (Y,T)$ is somewhat continuous and A is a dense subset of X and S_A is the induced topology for A then $f|_A:(A,S_A) \to (Y,T)$ is somewhat continuous.

Proof: Let B be a dense subset of (A,S_A). Since A is dense in X, B is a dense subset of X. Since f is somewhat continuous, by Theorem 3, $f(B)$ is dense in $f(X)$. But $f|_A(B) = f(B) \subset f(A) \subset f(X)$. Therefore $f|_A(B)$ is a dense subset of $f(A)$ and by Theorem 3, $f|_A$ is somewhat continuous.

Example 8. Let $X = \{a,b,c\}$. Let $S = \{\emptyset, X, \{a,b\}, \{c\}\}$ and $T = \{\emptyset, X, \{b,c\}\}$ be topologies for X. Let $A = \{a,b\}$. Then A is both an open and closed subset of (X,S). Let f be the identity from (X,S) onto (X,T). Then f is somewhat continuous, but its restriction to A is not somewhat continuous. Thus requiring the subset to be either open or closed is not sufficient to insure that the restriction is somewhat continuous.

Theorem 5. If (X,S) and (Y,T) are topological spaces and A is an open subset of X and $f:(A,S_A) \to (Y,T)$ is a somewhat continuous function such that $f(A)$ is dense in Y, then any extension F of f mapping (X,S) into (Y,T) is somewhat continuous.

Proof: Let F be an extension of f. Let $U \in T$ such that $F^{-1}(U) \neq \emptyset$. Then $U \neq \emptyset$ and since $f(A)$ is dense in Y, $U \cap f(A) \neq \emptyset$. Thus $f^{-1}(U) \neq \emptyset$. Therefore, there is a $V \in S_A$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$. But since A is open, $V \in S$ and since F is an extension of f, $V \subset f^{-1}(U) \subset F^{-1}(U)$. Hence F is somewhat continuous.

The next two examples show that neither A being open in X nor $f(A)$ being dense in Y can be omitted in the previous theorem.

Example 9. Let $X = \{a,b\}$. Let $S = \{\emptyset, X, \{a\}\}$ and $T = \{\emptyset, X, \{b\}\}$. Let $A = \{a\}$. Then $S_A = \{\emptyset, A\}$. Define $f:(A,S_A) \rightarrow (X,T)$ by $f(a) = a$. Then f is continuous and A is an open subset of X. The extension F of f defined by $F(a) = a$ and $F(b) = b$ is a function from (X,S) onto (X,T) which is not somewhat continuous.

Example 10. Let $X = \{a,b\}$. Let $S = \{\emptyset, X\}$ and $T = \{\emptyset, X, \{a\}\}$ be topologies for X. Let $A = \{a\}$. Define $f:(A,S_A) \rightarrow (X,T)$ by $f(a) = a$. Then $f(A)$ is dense in (X,T) and f is continuous. The extension F of f defined by $F(a) = a$ and $F(b) = b$ is a function from (X,S) onto (Y,T) which is not somewhat continuous.

CHAPTER III

WEAKLY EQUIVALENT TOPOLOGIES

Definition 10. If X is a set and T and T' are topologies for X, then T is said to be weakly equivalent to T' provided if $U \in T$ and $U \neq \emptyset$, then there is a $V \in T'$ such $V \neq \emptyset$ and $V \subset U$ and if $U \in T'$ and $U \neq \emptyset$, then there is a $V \in T$ such that $V \neq \emptyset$ and $V \subset U$.

Example 11. Let $X = [0,1]$. Let T be the usual topology for X. Let $T' = \{X, \emptyset, T - \{U/U \in T, 1/2 \in U\}\}$ be another topology for X. Then T and T' are weakly equivalent topologies such that (X,T) is a metric space, but (X,T') is not even a T_1 space.

Example 12. Let $X = (0,1)$. Let T be the usual topology for X. Let $T' = \{X, \emptyset, T - \{U/U \in T, 1/2 \in U\}\}$ be another topology for X. Then T and T' are weakly equivalent topologies such that (X,T') is a compact space, but (X,T) is not a compact space.

Example 13. Let $X = \{\text{Reals}\}$. Let T be the usual topology and T' be the upper limit topology. T and T' are weakly equivalent topologies such that (X,T) is a connected space, but (X,T') is not a connected space.

Theorem 6. If $f:(X,S) \rightarrow (Y,T)$ is somewhat continuous and S' is a topology for X which is weakly equivalent to S, then $f:(X,S') \rightarrow (Y,T)$ is somewhat continuous.

Proof: Let $U \in T$ such that $f^{-1}(U) \neq \emptyset$. Then since $f:(X,S) \rightarrow (Y,T)$ is somewhat continuous there exists a non-empty element V in S such that $V \subset f^{-1}(U)$. Since S and S' are weakly equivalent there exists a non-empty set R in S' such that $R \subset V \subset f^{-1}(U)$. Therefore $f:(X,S') \rightarrow (Y,T)$ is somewhat continuous.

<u>Theorem 7</u>. If $f:(X,S) \rightarrow (Y,T)$ is somewhat continuous function from X onto Y and S' and T' are topologies for X and Y respectively such that S' is weakly equivalent to S and T' is weakly equivalent to T, then $f:(X,S') \rightarrow (Y,T')$ is somewhat continuous.

Proof: Let $U' \in T'$ such that $f^{-1}(U') \neq \emptyset$. Then $U' \neq \emptyset$. Therefore there is a $U \in T$ such that $U \neq \emptyset$ and $U \subset U'$. Since $U \neq \emptyset$ and f is onto, $f^{-1}(U) \neq \emptyset$. Thus there is a $V \in S$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$. There is a $V' \in S'$ such that $V' \neq \emptyset$ and $V' \subset V$, thus $V' \subset f^{-1}(U')$ and $f:(X,S') \rightarrow (Y,T')$ is somewhat continuous.

<u>Theorem 8</u>. Suppose X is a set. If T and T' are weakly equivalent topologies for X and if T is separable, then T' is separable.

Proof: (X,T) contains a countably dense subset M. Let $U \neq \emptyset$, and $U \in T'$. Since T and T' are weakly equivalent, there is a non-empty set $V \in T$ such that $V \subset U$. Since M is dense in (X,T), $M \cap V \neq \emptyset$. Hence $M \cap U \neq \emptyset$. Thus T' is separable.

CHAPTER IV

SOMEWHAT OPEN FUNCTIONS

Definition 11. Let (X,S) and (Y,T) be topological spaces. A function $f:(X,S) \to (Y,T)$ is said to be somewhat open provided that if $U \in S$ and $U \neq \emptyset$, then there is some open set $V \in T$ such that $V \neq \emptyset$ and $V \subset f(U)$.

Example 14. Let $X = \{a,b,c\}$. Let $S = \{X,\emptyset,\{a,b\}\}$ and $T = \{X,\emptyset,\{b\},\{a,c\}\}$ be topologies for X. Let $f:(X,S) \to (X,T)$ be defined by $f(x) = x$. Then f is somewhat open but f is not open.

Remark 4. It is clear that every open function is somewhat open.

Theorem 9. If $f:(X,S) \to (Y,T)$ and $g:(Y,T) \to (Z,U)$ are somewhat open functions, then $g \circ f: (X,S) \to (Z,U)$ is somewhat open.

Proof: Let G be open in (X,S) with $G \neq \emptyset$. Since f is somewhat open, $f(G)$ contains an open set $V \neq \emptyset$ of (Y,T). Since g is somewhat open $g(V)$ contains an open set $k \neq \emptyset$ of (Z,U). Hence $K \subset g(V) \subset g(f(U))$. Therefore $g \circ f$ is a somewhat open function.

Theorem 10. If $f:(X,S) \to (Y,T)$ is a function, then the following two conditions are equivalent:

(1) f is somewhat open, and

(2) If M is a dense subset of Y, the $f^{-1}(M)$ is a dense subset of X.

Proof: (1) $\rightarrow$ (2) Suppose f is somewhat open. Let M be a dense subset of Y and let U be a non-empty open subset of X. Then there is a $V \in T$ such that $V \neq \emptyset$ and $V \subset f(U)$. Thus V contains a point p of M. Since $V \subset f(U)$, $p \in f(U)$ and $f^{-1}(p) \cap U \neq \emptyset$. Let $q \in f^{-1}(p) \cap U$. Since $f^{-1}(p) \subset f^{-1}(M)$, $q \in f^{-1}(M)$. Thus U contains a point of $f^{-1}(M)$ and $f^{-1}(M)$ is a dense subset of X.

(2) $\rightarrow$ (1) Suppose f has property (2). Let U be a non-empty open subset of X. Suppose $f(U)$ contains no non-empty open subset of Y. Then every non-empty open subset of Y intersects $Y - f(U)$ and thus $Y - f(U)$ is dense in Y. Therefore $f^{-1}(Y-f(U))$ is dense in X. Let $w \in f^{-1}(Y-f(U))$. Then $f(w) \in Y-f(U)$. Therefore $f(w) \notin f(U)$ and thus $w \notin U$. Therefore $w \in X - U$ and $f^{-1}(Y-f(U)) \subset X - U$. But U is a non-empty open subset of X and $f^{-1}(Y-f(U)$ is dense in X. Thus there is a $p \in U \cap f^{-1}(Y-f(U)) \subset U \cap (X-U)$ which is impossible. Hence f is somewhat open.

<u>Theorem 11</u>. If $f:(X,S) \rightarrow (Y,T)$ is one-to-one and onto then the following two conditions are equivalent:

(1) f is somewhat open and

(2) If C is a closed subset of X such that $f(C) \neq Y$ then there is a closed subset D of Y such that $D \neq Y$ and $D \supset f(C)$.

Proof: (1) $\rightarrow$ (2) Suppose f is somewhat open. Let C be a closed subset of X such that $f(C) \neq Y$. Since f is onto $C \neq X$. Thus $X - C$ is a non-empty open subset of X. Therefore there is a $V \in T$ such that $V \neq \emptyset$ and $V \subset f(X-C)$. Let $p \in f(C)$. Thus $f^{-1}(p) \in C$ so $f^{-1}(p) \notin X - C$. Therefore $p \notin f(X-C)$ and $p \notin V$. Thus $p \in Y - V$ and hence $Y - V \supset f(C)$. Since $V \neq \emptyset$, $Y - V \neq Y$.

(2) $\rightarrow$ (1) Suppose f has property (2). Let $U \in S$ such that $U \neq \emptyset$. Then $X - U \neq X$ and since f is one-to-one, $f(X-U) \neq Y$. Since $X - U$ is closed there is a closed subset D of Y such that $D \neq Y$ and $D \supset f(X-U)$. Let $p \in Y - D$. Then $p \notin D$ and thus $p \notin f(X-U)$. So $f^{-1}(p) \notin X - U$ and so $f^{-1}(p) \in U$. Thus $p = f(f^{-1}(p) \in f(U)$ and hence $Y - D \subset f(U)$. But $D \neq Y$ so $Y - D \neq \emptyset$. Thus f is somewhat open.

The following two examples show that neither one-to-one nor onto can be omitted in Theorem 12 for either direction.

Example 15. Let $X = \{a\}$, $Y = \{a,b\}$. Let $S = \{\emptyset, X\}$ be a topology for X. Let $T = \{\emptyset, Y, \{a\}\}$ and $U = \{\emptyset, Y, \{b\}\}$ be two topologies for Y. Define $f:X \rightarrow Y$ by $f(a) = a$. Then f is one-to-one, $f:(X,S) \rightarrow (Y,T)$ is somewhat open but does not fulfill (2) of Theorem 12, and $f:(X,S) \rightarrow (Y,U)$ fulfills (2) of Theorem 12, but is not somewhat open.

Example 16. Let $X = \{a,b,c\}, Y = \{a,b\}$. Let $S = \{X,\emptyset.\{a,b\}\}$ and $T = \{X,\emptyset,\{a\}\}$ be topologies for X. Let $U = \{Y,\emptyset\}$ be a topology for Y. Define $f:X \rightarrow Y$ by $f(a) = a$, $f(b) = b$, $f(c) = a$. Then f is onto, $f:(X,S) \rightarrow (Y,U)$ is somewhat open but does not fulfill (2) of Theorem 12 and $f:(X,T) \rightarrow (Y,U)$ fulfills (2) of Theorem 12, but is not somewhat open.

Theorem 12. If $f:(X,S) \rightarrow (Y,T)$ is a somewhat open function and S' and T' are topologies for X and Y respectfully such that S' is weakly equivalent to S and T' is weakly equivalent to T, then $f:(X,S') \rightarrow (Y,T')$ is somewhat open.

Proof: Let $U' \in S'$ such that $U' \neq \emptyset$. Then there is a $U \in S$ such that $U \neq \emptyset$ and $U \subset U'$. There is a $V \in T$ such that $V \neq \emptyset$ and $V \subset f(U)$. There is a $V' \in T'$ such that $V \neq \emptyset$ and $V' \subset V$. Then $V' \subset V \subset f(U) \subset f(U')$. Thus $f:(X,S') \rightarrow (Y,T')$ is somewhat open.

CHAPTER V

TOPOLOGICAL PROPERTIES THAT CARRY OVER UNDER OUR WEAKER FUNCTIONS

Theorem 13. If f is a somewhat continuous function from X onto Y and X is separable, then Y is separable.

Proof: Let M be a countable dense subset of X. Then $f(M)$ is countable and by Theorem 3, $f(M)$ is a dense subset of $f(X)$ which is Y. Thus Y is separable.

Definition 12. Let f be a function of X into Y. f is said to be a somewhat homeomorphism provided f is on-to-one, onto, somewhat continuous, and somewhat open.

Remark 4. It is clear that if f is a somewhat homeomorphism, then f^{-1} is a somewhat homeomorphism.

Theorem 14. Let f be a mapping from (X,S) into (Y,T). Let f be a somewhat homeomorphism. If A is a nowhere dense subset of X, then $f(A)$ is a nowhere dense subset of Y.

Proof: Let A be a nowhere dense subset of X. Let U be a non-empty element of T. Then $f^{-1}(U) \neq \emptyset$ and thus there is a $V \in S$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$. Since A is nowhere dense there is a $W \in S$ such that $W \neq \emptyset$, $W \subset V$, and $W \cap A = \emptyset$. There is a $U' \in T$ such that $U' \neq \emptyset$ and $U' \subset f(W)$. Since $W \cap A = \emptyset$ and f is on-to-one, $f(W) \cap f(A) = \emptyset$. Thus $U' \cap f(A) = \emptyset$. Now $U' \subset f(W) \subset f(V) \subset f(f^{-1}(U)) = U$. Thus $f(A)$ is a nowhere dense subset of Y.

Corollary 14.1. Let $f:(X,S) \rightarrow (Y,T)$ be a somewhat homeomorphism. If (X,S) is of first category (second category) then (Y,T) is of first category (resp second category). This corollary follows immediately from Theorem 15.

Theorem 15. If (X,T) is a topological space and T' is a topology for X which is weakly equivalent to T then (X,T) is a Baire space (first category)(second category) if and only if (X,T') is a Baire space (resp first category)(resp second category).

Proof: From previous comments the identity function from (X,T) onto (X,T') is a somewhat homeomorphism and thus the result follows from 2. coro 1, page 383 and Corollary 15.1.

Definition 13. A topological space (X,T) is said to be a D-space provided every non-empty open subset of X is dense in X.

Theorem 16. Let f be a mapping of (X,S) into (Y,T). If f is a somewhat continuous function from X onto Y and X is a D-space, then Y is a D-space.

Proof: Let U be a non-empty open subset of Y. Since f is onto, $f^{-1}(U) \neq \emptyset$. Since f is somewhat continuous there exists a $V \in S$ such that $V \neq \emptyset$, and $V \subset f^{-1}(U)$. Since X is a D-space, V is dense in X. Therefore by Theorem 3, $f(V)$ is a dense subset of $f(X) = Y$. Since $V \subset f^{-1}(U)$, $f(V) \subset U$ and thus U is a dense subset of Y. Hence Y is a D-space.

<u>Theorem 17</u>. Suppose $f:X \to Y$ is a continuous, somewhat open function from X onto Y. If X is locally compact on a dense subset of X(i.e. there is a dense subset D of X such that if $p \in D$, then there is an open subset U of X containing p such that $\overline{U}$ is compact) and Y is Hausdorff, then Y is locally compact on a dense subset of Y.

Proof: Let U be a non-empty open subset of Y. We must find a point of U at which Y is locally compact. Since f is continuous and onto, $f^{-1}(U) \neq \emptyset$ and $f^{-1}(U)$ is an open subset of X. Since X is locally compact on a dense subset of X, there is an element $x \in f^{-1}(U)$ such that X is locally compact at x. Thus we can find an open subset W of X containing x and contained in $f^{-1}(U)$ such that $\overline{W}$ is compact. Since W is a non-empty open subset X and f is somewhat open, there is a non-empty open subset V of Y such that $V \subset f(\overline{W})$. Let $z \in V$. Then $z \in U$. Since Y is Hausdorff and $f(\overline{W})$ is compact, $f(\overline{W})$ is closed. Thus $\overline{V} \subset f(\overline{W})$. Hence $\overline{V}$ is compact and thus Y is locally compact at z which is a point of U.

The following example shows that the continuous, somewhat open image of a locally compact space need not be locally compact everywhere.

Example 17. Let X = Reals - $\{$non-zero integers$\}$. Let S be the induced topology for X gotten from the usual topology for the reals, and let T be the topology for X which has a base $(S-\{V/0 \in V, V \in S\}) \cup \{[(-\infty,-r) \cup (-1/r,1/r) \cup (r,\infty)] \cap X/r > 0\}$. Then (X,S) is locally compact and (X,T) is Hausdorff. The identity function from (X,S) onto (X,T) is continuous and somewhat open. Yet (S,T) is not locally compact at 0.

SUMMARY

This paper is concerned with an investigation of some of the recent definitions of weaker forms of continuity. Special attention has been given to the definition of a somewhat continuous function. In Chapter V, the following theorem is proved: Suppose $f:X \to Y$ is a continuous, somewhat open function from X onto Y. If X is locally compact on a dense subset of X and Y is Hausdorff, then Y is locally compact on a dense subset of Y.

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